A MYSTERY DEMYSTIFIED: THE CONNECTION BETWEEN MATHEMATICS AND PHYSICS

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The enigma: the "unreasonable effectiveness" of mathematics in natural science

Mathematics is an understanding of nature emptying it out of all particularity and temporality: a view of nature without either individual phenomena or time. It empties nature out of them the better to focus on one aspect of reality: the recurrence of certain ways in which pieces of the world connect with other pieces. Its subject matter are the structured wholes and bundles of relations that outside mathematics we see embodied only in the time-bound particulars of the manifest world.

The distinctiveness of the mathematical perspective -- its evisceration of particularity and its suppression of time -- helps explain the power of mathematics to illuminate a universe in which time holds sway and particularity is everywhere. This power, nevertheless, perpetually subjects us to a twofold risk.

The first risk is to mistake the mathematical representation of a slice of the one real world -- the slice that has to do with bundles of relations and with structured wholes -- for privileged, indubitable insight into a separate, nature-transcending realm of mathematical truths. There is no such realm, any more than there is a multitude of unobservable universes (now commonly called the multiverse) whose existence we postulate only because they fill the otherwise empty boxes of a mathematical conception, disguised as a physical theory.

The second risk is that we allow ourselves to be lulled by the effectiveness and beauty of our mathematical propositions into the belief that nature shares in their timelessness. It would do so, most convincingly, by operating under the force of eternal laws and symmetries. Such regularities achieve adequate expression only when they can be represented mathematically. Their susceptibility to mathematical representation confirms, according to this illusion, their claim to participate in the freedom of mathematics from time. It does not. To believe that it does is to spoil the gift of mathematics to physics.

If we put aside the technical disputes of the contemporary philosophy of mathematics, we can see that almost all the known options in our philosophical understanding of mathematics fall into two families of ideas.

According to the first family of ideas, mathematics is discovery. It is the progressive (or recollected) discovery of the truths that exist in a domain of mathematical facts uncompromised by the vicissitudes and the variations of the manifest world.

According to the second family of ideas, mathematics is invention: the free development of a series of conventions of quantitative and spatial reasoning. This conventional practice of analysis may be rule-guided or even rule-bound, but the rules are themselves inventions. There is no closed list of motives for this inventive practice. Some have little or nothing to do with the deployment of mathematical analysis in natural science. Others take this deployment as their goal.

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For the discovery theories, there is a fact of the matter in every true mathematical question, although it is not a fact about or in the natural world. For the invention theories, there can be no fact of the matter. We who invented mathematical propositions remain their arbiters.

Both families of ideas about the nature and applicability of mathematics suffer from a common defect. They fail to account for why mathematics has been as useful to science as it has. They fail to make adequate sense of what Eugene Wigner called "the unreasonable effectiveness of mathematics" in the scientific study of nature.

The discovery views fail to do so by setting up a discontinuity between natural fact and mathematical truth for which they then neglect to provide any bridge. A metaphysics like Plato's would, if it could be believed, build such a bridge. It would do so in the form of an account of how natural phenomena come to share, albeit dimly and imperfectly, in the fuller, deeper reality of mathematical truths. Ever since Plato, however, most who have been attracted to the discovery views have balked at any such ontology. In its absence, they must contend with the metaphysical dualism -- natural history and particularity on one side, timeless and faceless truth on the other -- that their position implies.

The invention views fail to account for the applicability of mathematics in natural science by making its applicability appear to be either a happy accident or an abstract engineering. It would be happy accident if it just turned out -- mysteriously -- that so many of our mathematical inventions, undertaken for other reasons, supplied what each succeeding wave of theories in physics happened to need. It would be abstract engineering if we could always rely on making up, after the fact, the mathematical theories that our physical conjectures require. If mathematics were so elastic that we could be sure of getting from it, by invention or construction, whatever we need formally to represent our causal ideas about the workings of nature, it would fall under suspicion. The ease with which we could get from it whatever we want would rob it, by the same token, of the power to provide any independent check on our theory-making in natural science.

Neither happy accident nor abstract engineering do justice to the historical interactions between mathematics and physics: as now mathematics, and now science, advance ahead of the other. Neither thesis takes adequate account of the interplay between the two great forces that have influenced the history of mathematics: the internal development of mathematical reasoning (as each breakthrough generates a new set of problems) and the provocation offered by the development of natural science (when the available mathematical tools prove inadequate to the advancement of novel physical theories or to the refinement of disconcerting physical intuitions).

What mathematics is

Among the defining attributes of mathematics are explication, recursive reasoning, and fertility in the making of equivalent propositions. They are connected, and they overlap.

Explication is the working out of what is implied in a particular conception of a structured whole or of a bundle of relations: not just any structured whole or bundle of relations, but one foreign to the natural experience of time-bound particulars. Once the world is robbed of its flesh -- the flesh of the particulars that begin, move, and end in time -- and reduced to the skeletal form of its most general traits, it continues to have, in this skeletal mode, structure or content.

This content is described by mathematical conceptions. Each mathematical idea refers to a piece of the residual structure; each serves as a summary reference. Explication is the progressive development of the mathematical propositions that are implied in the summary reference and that depict the relevant piece of the world without flesh.

A second attribute of mathematics is its reliance on recursive reasoning. Reasoning is recursive when it takes itself for a subject, or, more precisely, when it applies to the procedures that it deploys. Recursive reasoning enables us to pass from enumerations to generalizations; we jump off from the particular to the general by suggesting the general rule implicit in what, up till then, had seemed to be a mere enumeration of particulars. (This is the aspect of recursive reasoning that Charles Sanders Peirce called abduction, the better to emphasize its contrast with induction, for which it is commonly mistaken.) It allows us to reach strong and rich conclusions on the basis of weak and parsimonious assumptions.

Recursive mathematical reasoning enables us to develop our insight into structured wholes and bundles of relations indirectly. It does not by observing nature or even analyzing the presuppositions or implications of theories in natural science. Instead, it does so by generalizing mathematical ideas used to explore more particular bundles and wholes. It is bootstrapping. Nevertheless, it is bootstrapping of an activity that is turned outward to nature, viewed under a particular aspect. Its proximate subject matter is mathematical reasoning itself. Its ulterior subject matter is the eviscerated natural world – hollowed out of time-bound particulars – with which mathematics deals, when it is not dealing with itself.

A third attribute of mathematics is its fecundity in the statement of equivalent propositions. A major part of mathematical reasoning consists in showing how one line of analysis can be restated in terms of another. The practical importance of this feature of mathematics for natural science is manifest in the vital role that gauge symmetries play in physics and cosmology.

Just as explication can be mistaken for a superstitious conceptualism and recursive reasoning for induction, so can the multiplication of equivalent propositions be mistaken for the marking of synonymy. It is as if we already understood the truth about the aspect of the world that is studied and mathematics and wanted only to organize better the mathematical language in which to represent this achieved understanding. We would organize the language of mathematics better, according to this misinterpretation of the facility for equivalence, by clarifying which combinations of symbols are and are not synonymous.

The basis and nature of this third trait of mathematics lie in another direction. The abstraction of mathematics exposes it to a danger to which natural science has other antidotes. It is the danger of failing to distinguish the ordered wholes and sets of relations it studies from their conventional expressions. (Those who embrace the view of mathematics as simple invention take succumbing to this danger as their program.) Precisely because in mathematics we lack the manifest, time-bound world to surprise, baffle, and correct us, we must try, at every turn, to distinguish our ideas about nature, abstracted, eviscerated of particulars, from their conventional expressions. The best way to do so is to insist on restating the ideas in equivalent forms -- in alternative conventions.

These three atttributes fail adequately to account for the distinctive nature and the special power of mathematics. Not only are they insufficient to account for mathematics; they are also unable to account for themselves.

Our mathematical and logical reasoning has a characteristic that places it in sharp contrast to our causal explanations. A cause comes before an effect. Causal explanations make no sense outside time; causal connections can exist only in time. However, the moves in a mathematical or logical chain of argument do occur outside time. To take a simple example, there is no temporal succession -- no real-world before and after -- in the relation between the conclusion of a syllogism and its major and minor premises. They are not so much simultaneous, as they are outside the realm of time altogether.

In nature, timeliness and particularity are tightly linked. Every particular in the world exists in time. Everything that exists in time is a particular, although the nature of particularity – that is to say, how particulars are distinguished from one another – is itself subject to change in the course of time. The manifest world is a world of particulars as well as a world of time. Even the angels, to be able to intervene in historical time, require distinct personality, despite being said to lack bodies.

That the relations among mathematical propositions exist outside time is a fact consistent with two other truths that might incorrectly be thought to contradict this claim. The first such truth is that mathematical and logical reasoning takes place in time. Its temporal enactment in the minds of time-bound individuals says nothing about its content. The propositional content is one thing; the psychological phenomenon, another.

The second such truth, more remarkable and puzzling than the first, is that a form of discourse that is not temporal can be used to describe movement and change in time. The most striking example is the single most important instance of the applicability of mathematics in natural science: the work that the calculus does in mechanics. The calculus is used -- indeed in part in was invented by Newton and Leibniz -- to furnish a mathematical representation of movement and change in a configuration space limited by initial conditions (and therefore representing a part of the world rather than the world as whole) and governed by supposedly immutable laws. The seemingly paradoxical use of non-temporal connections to represent changes in time has ceased to mark physics only insofar as physics despairs of the attempt to explain change and puts structural analysis in the place of causal explanation.

The use of statements about connections outside time to represent phenomena in time is not just an aspect among others of the problem of the "unreasonable effectiveness of mathematics." It is the kernel of that enigma. The enigma has an explanation, in fact two explanations: one, psychological and evolutionary; the other, methodological and metaphysical. Before exploring them, however, I must go further in depicting the war that mathematics wages against time.

Mathematics deals with nature as well as with itself. However, it addresses a nature from which time and, together with time, all phenomenal distinction have been sucked out. The world that it represents is neither the real one nor another one -- a domain of timeless mathematic objects. It is the real world -- the only one that exists -- robbed of time. The subject matter of mathematics is a visionary simulacrum of the one real world. Unlike the real thing, the simulacrum is shadowy and timeless. It is preserved against corruption and change only because it is removed from nature, in which time and particularity rule.

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The enigma solved: the effectiveness of mathematics
in physics is reasonable because it is relative

How can the analysis of such a proxy for the world prove so useful to the representation of causal relations in the real thing? Consider the explanatory advantages of doing what mathematic does, which is to deal with structured wholes and bundles of relations without regard to natural time and natural distinctions.

We cannot set aside the particularity of phenomena without also depriving them of their temporality: everything in the natural world is sunk in time. However, it is only by disregarding both time and natural distinction that we can deal with relations and combinations in their most general form. We can then more easily form ideas and inferences about these general connections, free from the spell of the embodied, particular, and temporal forms in which we encountered them. We can use such inferences and ideas to explain change and movement in time. They provide natural science with what it could never achieve if its imagination of the possibilities of connection were limited to the forms in which we meet them through our senses or our instruments.

But what exactly is it that we address when we deal with the simulacrum -- the world without time or particularity? This is the fourth attribute of mathematics: the subject matter distinctive to a way of thinking that sucks the world dry of time.

A simple but incomplete answer is that we are left with space and with number, or more precisely with the connection between number and space. A more adequate and comprehensive answer is that we are left with the most general relations among parts of the world: structured wholes and bundles of relations. In mathematics, we deal with them in their incorporeal and therefore timeless form. It is then only by exercising a self-critical vigilance that we avoid the unwarranted supposition that if we are able to conceive and to reason about them in this abstracted form, and if in reasoning about them we experience a constraint that our axioms and rules of inference seem insufficient to explain, they must be objects of a special type, inhabiting a distinct part of reality. We turn a practice into an ontology: a metaphyical view of a permanent stock of abstract objects, which supposedly provide both mathematics and physics with their ultimate subject matter.

The referents of our mathematical notions, however, are not a distinct realm of objects. They are the very same objects of the one real world, viewed from the vantage point of a special way of thinking, one that is blind to the phenomenal distinctions and to the temporal variations of nature.

The numerical relations that we discover, such as the distribution of prime numbers, are not facts about a realm of mathematical objects. They are facts about the only world that there is, although they are, so to speak, second-order facts. They are not facts about the interactions among the natural phenomena that make up nature. They are facts about the second-order or higher-level relations among the ways of numbering, counting, or quantifying that are suggested to us by such interactions.

They are neither facts about a world other than ours, nor made-up, free constructions. They are facts about this one real world, although facts about what this world looks like to us at one order of remove. This order of remove does not save them from having the same startling, it-just-happens-to-be-that way character of ordinary natural facts.

An implication of this view is that there is a basic asymmetry in the relation of mathematics to space and to time. Its relation to space is intimate and internally connected with

number. Its relation to time is distant and external, even when movement and change in time are (as in the original applications of the calculus) the very subject that the mathematical reasoning is used to represent.

The history of twentieth-century physics shows the influence of this asymmetry. Even when it proposed to connect space and time and to deny that spacetime could be understood as an absolute and invariant background to physical phenomena, twentieth-century physics spatialized time rather than temporalizing space. The use of a spatial metaphor to describe time as a "fourth dimension" is a crude manifestation of this bias. Its most significant expression, however, is persistent equivocation about the reality of time (unaccompanied by any such wavering about space). One of the results was and is the reaffirmation, mostly unthinking but sometimes considered, of the idea of a timeless framework of natural laws underwriting our causal explanations of nature.

If mathematics exhibited only the first three of these four characteristics -- its practice of explication, its devotion to recursive reasoning, and its fertility in equivalent propositions -- we might be justified in taking something from the view of mathematics as discovery and something from the view of mathematics as invention, and in using each of these approaches to make up for the inadequacy of the other. In this undertaking, however, we would have missed the most decisive and astonishing feature of mathematics, and the one by virtue of which it can be neither invention nor discovery, as they are conventionally understood. This trait -- the fourth attribute of mathematics -- is the study of a counterfeit version of the world, of the only world that there is: a version from which the flesh and flow of things have been banished.

Tempting illusions

The view invoking these four attributes of mathematics helps explain and resist the two temptations to which our mathematical capabilities subject us. Those temptations form part of the price that we pay for these powers.

The first temptation is to imagine that our faculties of mathematical abstraction give us access to a doubly privileged form of insight. It would be privileged by virtue of addressing a realm of timeless truth, distinct from the natural world in which we move. It would also be privileged as the result of enjoying a species of certainty for which we dare not hope in the practice of natural science. In this light, mathematics begins to seem not just like higher insight into this our world but also like insight into a higher world.

The second temptation is to deny or to discount the reality of time, given that the shadowy version of the world with which mathematics presents us is timeless. Our mathematical and logical reasoning is a fifth column within the mind, working against recognition of the inclusive reality of time. In the conflict between what nature seems to be -- steeped in time as well as endlessly varied -- and what mathematics appears to say about reality, we may be seduced into siding with mathematics against nature. (It is more than a hypothetical danger. The central tradition of modern physics, from Galileo and Newton to quantam mechanics and relativity, received from mathematics a poisoned chalice: the idea of timeless laws and of an immutable, ultimate structure of nature – but that is another, longer argument.)

The two temptations threaten us with different degrees of deception: the second more dangerous than the first. The relation between them is asymmetrical. We can succumb to the first

without giving in to the second. However, it is difficult to surrender to the latter without having accepted some of the former. We are led to diminish or even to reject the reality of time by the conviction that the most reliable truths, the truths of mathematics, are timeless.

The study of nature, unblinkered by metaphsyical prejudice and open to the surprises of experience and experiment, gives us reason to reject both temptations. It presents us with a world that contains both less and more than any catalog of mathematical ideas. The world contans less than mathematics does because the most important fact about the universe is that is what it is rather than something else. Mathematics suggests many ways in which nature might be organized but is not in fact organized, so far as we can observe. Only an intellectual abdication, subversive of the discipline of natural science, could prompt us to believe that such unobserved mathematical connections must be realized in unobservable universes. The world contains more than mathematics does because everything in nature changes, including change itself, and because nature is full of singular events and phenomena.

Mathematics cannot replace physical insight

That one mathematical connection rather than another strikes one of nature's chords always has physical reasons: reasons that can be formulated in non-mathematical language. Mathematics may help suggest a physical picture. It can never, on its own, either lead us to physical truth or suffice to our understanding of such truth. We do not overcome the limitations of a scientific idea simply by giving it mathematical expression.

In the history of physics no example of the supposedly preternatural power of mathematics to lift the veil of nature is more striking than Newton's inverse square law of gravitation, according to which the gravitational force connecting two bodies varies in direct proportion to the product of their inertial masses and in inverse proportion to the square of the distance between then. Why the square of the distance rather than some other, less simple and pleasing measure? Why the neat and disconcerting symmetry? And why does an inverse square rule apply to a number of physical phenomena other than gravitation?

Newton's inverse square law accords with a visual representation that we have independent, physical reasons to take as accurate, appealing to conserved lines of force, or flux lines. Imagine the gravitational field of the Sun to be represented by lines fanning outward from the center of the Sun, always in the radial direction. Suppose further the existence of a planet moving in a circular orbit at a distance d from the Sun's center.

Assume that the force felt by the planet is proportional to the density of the field lines at the distance d. The density falls off as $1/d^2$. It falls off at this rate because the number of field lines intersecting a sphere of any radius around the Sun remains constant. The density, which is the number divided by the area of the sphere, diminishes according to this area, which is d^2. Consequently, the force weakens at the measure of $1/d^2$. Only if the force falls off according to the rule 1/distance-squared will the picture of lines of force that I earlier invoked hold.

An instance of the same way of thinking, not directly related to gravity, suggests the generality of the spatial reasons for its wide application. Compute the dot product of the force field with the unit normal to the surface and integrate the resulting function over the surface. The result is proportional to the mass of the interior bounded by the surface. This relation recurs no matter how the mass is distributed.

The same mathematics works for the electric force, with electric charge in the place of mass. It works as well in a number of other contexts, for reasons that have to do with the regular and recurrent spatial disposition of natural forces in the established universe.

The less we grasp the non-mathematical reasons for the application of mathematics (and in each of these examples we understand them only very incompletely), the more enigmatic and disconcerting the application of mathematics will appear to be. We will be tempted to bow down to mathematics as the custodian of nature's secrets. That the laws of parts of nature are written as mathematical equations then bewitches us into believing that all the workings of all of nature are foreshadowed in the truths of mathematics.

A study of the history of mathematics can help save us from these illusions. Having departed initially from our direct experience of the world through its extirpation of time and phenomenal particularity, mathematics moves yet further away from it thanks to the overriding influence of internal development: the way in which line of mathematical reasoning may inspire another. Only its engagement with science qualifies this influence, as science comes to play for mathematics the role that our sensible experience of nature once performed. Nevertheless, by the paradox that characterizes the entire history of mathematics, this removal from the world of the senses, only increases the power of mathematics to suggest news ways of thinking about how parts of reality may connect.

It follows from this interpretation of the history of mathematics that any given mathematical construction will have no assured application to the one real world. The price that mathematics pays for the enhancement of its power through internal development is the loss of any guarantee that its ideas will find application in the study of nature. Some will, and some will not. Having fallen short of the world (through its exclusion of time and particularity), its ideas will also overshoot the world. They will be both too little and too much to hold a mirror to reality. We shall have to dispense with the hope of a pre-established harmony between nature and mathematics.

A realistic and deflationary view of the role of mathematics in physics

What results is a view recognizing the unmatched powers and the unique perspective of mathematics. It nevertheless repudiates the Pythagorean claim, made on behalf of mathematics for at least twenty-six hundred years, that mathematical insight offers a shortcut to eternal truth about incorruptible objects. It sees mathematical reasoning as inquiry into the world -- the only world that there is, the world of time and fuzzy distinction -- only at one step of remove.

By an apparent paradox that goes to the heart of what is most interesting about mathematics, the distancing of mathematics from time and phenomenal variation, helps explain its power to support science in the investigation of a world from which phenomenal variation and time can never be expunged. The denial of these intrinsic features of nature turns out to be the condition for the development, in geometrical or numerical language, of our most general ideas about the ways in which pieces of reality connect.

The apparent paradox is so disconcerting in its content and so far-reaching in its implications that it seduces us into mistaking its significance. It inspires a dream: that mathematical insight provides a way out from the limitation of the senses, even as that limitation

is loosened by our scientific instruments. We may then fail to recognize that, for the scientist, mathematics is a good servant but a bad master.

The "unreasonable effectiveness of mathematics" is subject to a natural explanation, which dispels the appearance of unreasonableness in two convergent ways. The explanation does so in one way by showing the cognitive advantages that the timeless abstractions of mathematics can have for inquiry into a world drenched in time and full of particular phenomena and events. It does so in another way by suggesting that insofar as mathematical ideas are only obliquely connected with the phenomenal world studied by science -- extending what our senses and instruments allow us to see -- there is no guarantee that they will be applicable in natural science. They may or may not be. There is no pre-established harmony between physical intuition, or experimental discovery, and mathematical representation. Mathematics gives us no royal road to truth about nature. It allows us no exemption from struggling with the limits to our powers of observation and experiment.

What many have mistaken for an escape from ourselves turns out to be a road back into time, nature, and humanity.